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A Reynolds-uniform numerical method for the Prandtl solution and its derivatives for stagnation line flow

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SUMMARY

Farrell *et al.* (*Robust Computational Techniques for Boundary Layers*. Chapman & Hall/CRC: Boca Raton, 2000) develop a Reynolds-uniform numerical method for the solution of the Prandtl equations in the case of flow past a flat plate. In this paper, we examine the applicability of this Prandtl method to the stagnation line flow problem in a domain that includes the stagnation line. Stagnation line flow has been chosen because of its self-similar nature; reference solutions that approximate the exact solution of the problem to high levels of accuracy can be numerically obtained, allowing the error in the numerical approximations generated by the Prandtl method to be calculated. We present detailed results which verify that the method is Reynolds uniform. Global Reynolds-uniform error bounds are constructed for the numerical approximations to the velocity components and their scaled first derivatives, and the practical uses of these bounds are discussed. We show that the number of iterations required for convergence of this iterative method is Reynolds uniform. In addition, we test an experimental technique for computing global Reynolds-uniform error bounds, which can be used when solving flow problems for which no exact or reference solution is available. Experimental error bounds are constructed using this technique and are shown to be realistic upper bounds for the error values obtained with the use of the reference solutions. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: stagnation line flow; parabolic boundary layer; fitted mesh finite difference method; global reynolds-uniform error bounds; controllable accuracy; dimensionless shearing stress

1. INTRODUCTION

Consider the laminar flow of incompressible fluid past a solid body for large values of the Reynolds number *Re*. A thin boundary layer, characterized by a large velocity gradient across

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the flow, will exist at the surface of the body. The use of a numerical method that successfully resolves the boundary layer is necessary for the accurate computation of important physical quantities for such a system, for example, drag and heat transfer. Furthermore, it is desirable to have a single method that models the boundary layer flow uniformly well for a wide range of Re. The failure of standard numerical methods to satisfy these requirements has led to the development of Re or ε-uniform methods, where $\varepsilon = Re^{-1}$. Summaries of recent progress in this field can be found in References [1, 2].

The definition of an ε-uniform method states that an error bound of the form

$$\max_{0<\varepsilon\leqslant 1} \bar{E}^{\mathrm{ex}}_{\varepsilon,N} \leqslant \bar{C}_{\bar{p}} N^{-\bar{p}} \tag{1}$$

can be constructed for each component of the numerical solutions generated by the method, where N is the discretization parameter, $\bar{C}_{\bar{p}}$ and $\bar{p}$ are positive numbers that are independent of ε and N, and for each value of ε and N, $\bar{E}^{\mathrm{ex}}_{\varepsilon,N}$ is the exact error in the particular numerical solution component measured in the global maximum norm over the solution domain. In addition, the numerical solutions generated by an ε-uniform method are computable with an ε-uniform amount of work. $\bar{C}_{\bar{p}}$ and $\bar{p}$ are called the global ε-uniform error constant and order of convergence, respectively. The numerical solutions are said to be ε-uniformly convergent. A stronger definition of an ε-uniform method states that the errors in the appropriately scaled discrete first derivatives of the numerical solutions can also be bounded in the same way. The practical implications of these definitions include the following:

- The number of iterations required for convergence of an ε-uniform method applied to a non-linear system of equations is independent of the value of ε.
- For an ε-uniform method, error bound information can be used to determine the minimum value of N that must be used to generate numerical solutions and their scaled discrete first derivatives with a required level of guaranteed accuracy for all values of ε satisfying $0<\varepsilon\leqslant 1$.

Such properties are often needed in real computational fluid dynamics applications.

Farrell *et al.* in Reference [1] construct an iterative ε-uniform numerical method for the solution of the Prandtl boundary layer equations in the case of flow past a flat plate. In this paper, we examine the applicability of this Prandtl method to the stagnation line flow problem in a domain that includes the stagnation line. Stagnation line flow has been chosen because of its self-similar nature, which allows the Prandtl equations to be reduced to a single non-linear ordinary differential equation by the application of a simple set of transformations. The ordinary differential equation is a Falkner–Skan equation [3] and we refer to the transformed problem as the Falkner–Skan problem. The Falkner–Skan problem can be numerically solved using an ε-uniform method and the numerical solutions used to construct *reference* solutions that approximate the exact solution of the Prandtl stagnation line flow problem to *any* required level of accuracy. Previous work detailing the construction of such reference solutions, which we call the computed Falkner–Skan solutions of the Prandtl problem, can be found in Reference [4]. A computed Falkner–Skan solution of the Prandtl problem with a sufficiently high level of accuracy is employed to compute the errors in the numerical approximations generated by the Prandtl method. These error values are then used to verify that the Prandtl method is ε-uniform through the construction of global ε-uniform error bounds of form (1). We further apply the error values to test an experimental method for the calculation of global

ε-uniform error bounds. Such a technique is required when solving problems for which no exact or reference solution is available.

This paper is organized as follows. In Section 2 the stagnation line flow problem is described and mathematically defined. We outline the ε-uniform numerical method for the solution of this problem in Sections 3 and 4 the numerical results are presented and analysed. We conclude with a brief discussion in Section 5.

2. PROBLEM FORMULATION

Consider the plane laminar flow $(u, v, 0)$ of an incompressible fluid past a cylindrical body that is oriented at right angles to the xy-plane. Let the flow exhibit symmetry about the yz-plane. A stagnation line[¶] exists where the plane of symmetry intersects the front of the cylinder; we shall let this coincide with the z-axis. The stagnation line flow problem is an approximate model of the flow in the neighbourhood of the stagnation line, formed by replacing the curved surface in this region by a flat plate. The flat plate lies at $y = 0$.

For small values of ε a thin boundary layer of parabolic type is present in the stagnation line flow solution at the surface of the plate, and the dimensionless Prandtl boundary layer equations

$$\begin{aligned} -\varepsilon \frac{\partial^2 u_\varepsilon}{\partial y^2} + u_\varepsilon \frac{\partial u_\varepsilon}{\partial x} + v_\varepsilon \frac{\partial u_\varepsilon}{\partial y} &= u_0 \frac{\mathrm{d}u_0}{\mathrm{d}x} \\ \frac{\partial u_\varepsilon}{\partial x} + \frac{\partial v_\varepsilon}{\partial y} &= 0 \end{aligned} \tag{2}$$

adequately describe the flow. The use of the subscript ε signifies the dependence of the flow variables on the value of ε; similarly the subscript 0 denotes an inviscid flow variable. For this problem, the inviscid flow solution is defined by

$$(u_0, v_0) = (x, -y) \tag{3}$$

Note that the two-dimensional nature of the equations in (2) reflects the invariance of stagnation line flow in the z-direction.

By symmetry it suffices to study the stagnation line flow problem in the quarter plane $\{(x, y) : x \geqslant 0, y \geqslant 0\}$ for any value of z. An appropriate domain is $\bar{\Omega} = [0, 1] \times [0, 1]$. To solve the parabolic Prandtl equations, for which x plays the role of time, in $\bar{\Omega}$, we require an initial condition for u_ε at $x = 0$, boundary conditions for u_ε at $y = 0$ and $y = 1$, and a single boundary condition for v_ε at $y = 0$. The symmetry of the problem about $x = 0$ provides the initial condition $u_\varepsilon = 0$, and $u_\varepsilon = v_\varepsilon = 0$ at the plate $y = 0$ by the no-slip condition. Values of u_ε at $y = 1$ are given by the component $\bar{U}^{\mathrm{FS}}_{\varepsilon, N_{\mathrm{FS}}}$ of the Falkner–Skan solution of the Prandtl problem, where N_{FS} is the discretization parameter of the numerical method used to solve the Falkner–Skan problem (for further details, see Reference [4]). Thus, introducing the notation Γ_{B}, Γ_{L}, Γ_{T} for the bottom, left and top sides of $\bar{\Omega}$, respectively, we can write the Prandtl

[¶]Line along which $\mathbf{u} = 0$ in inviscid flow theory.

stagnation line flow problem in the form

$$(P_\varepsilon)\begin{cases}\text{Find } (u_\varepsilon, v_\varepsilon) \text{ such that for all } (x,y)\in\Omega\\ -\varepsilon\dfrac{\partial^2 u_\varepsilon}{\partial y^2}+u_\varepsilon\dfrac{\partial u_\varepsilon}{\partial x}+v_\varepsilon\dfrac{\partial u_\varepsilon}{\partial y}=x\\ \dfrac{\partial u_\varepsilon}{\partial x}+\dfrac{\partial v_\varepsilon}{\partial y}=0\\ \text{with the boundary conditions}\\ u_\varepsilon=0 \text{ on } \Gamma_{\mathrm{L}}\\ u_\varepsilon=v_\varepsilon=0 \text{ on } \Gamma_{\mathrm{B}};\quad u_\varepsilon=\bar{U}^{\mathrm{FS}}_{\varepsilon,N_{\mathrm{FS}}} \text{ on } \Gamma_{\mathrm{T}}\end{cases} \tag{4}$$

where $0<\varepsilon\leqslant 1$ and $\Omega=(0,1]\times(0,1)$ is the open domain corresponding to $\bar{\Omega}$. The successful resolution of the rapidly changing solution of (P_ε) within the boundary layer region requires a special numerical method; such a method is described in the next section.

3. NUMERICAL METHOD

It has been shown in Reference [5] that the use of a fitted mesh is essential for the construction of an ε-uniform numerical method for the solution of the Prandtl flat plate problem. In the absence of theoretical results, a similar approach is assumed necessary in the case of the Prandtl stagnation line flow problem (P_ε). We present the method of Farrell *et al.* [1] which consists of a classical upwind finite-difference scheme on a piecewise-uniform fitted mesh. Slight adaptations have been made for application of the method to (P_ε).

3.1. Fitted mesh

We begin by defining the piecewise-uniform mesh $\bar{\Omega}_{\varepsilon,N}$ to be the tensor product of one-dimensional meshes in the x- and y-directions

$$\begin{aligned}\bar{\Omega}_{\varepsilon,N}&=\bar{\Omega}^x_N\times\bar{\Omega}^y_{\varepsilon,N}\\ &=\{x_i: 0\leqslant i\leqslant N\}\times\{y_j: 0\leqslant j\leqslant N\}\end{aligned}$$

$\bar{\Omega}^x_N$ is taken to be uniform as the solution of (P_ε) exhibits smooth behaviour in the x-direction for all values of ε satisfying $0<\varepsilon\leqslant 1$. Thus,

$$x_i=ih_x \quad \text{for } 0\leqslant i\leqslant N$$

where

$$h_x=1/N$$

$\bar{\Omega}^y_{\varepsilon,N}$ is a piecewise-uniform mesh that condenses in the neighbourhood of the plate for small values of ε so as to resolve the boundary layer in the solution of (P_ε). We partition the line

$[y_0 = 0, y_N = 1]$ into two segments $[0, \sigma_y]$ and $[\sigma_y, 1]$, where σ_y represents the width of the boundary layer in numerical terms and is defined by

$$\sigma_y = \min\{1/2, C\varepsilon^{1/2} \ln N\} \tag{5}$$

C is an arbitrary positive constant. Each segment is further partitioned into $N/2$ uniform intervals to produce the piecewise-uniform distribution of points

$$y_j = \begin{cases} jh_y & \text{for } 0 \leqslant j \leqslant N/2 \\ \sigma_y + (j - N/2)H_y & \text{for } N/2 < j \leqslant N \end{cases}$$

where

$$h_y = 2\sigma_y/N; \quad H_y = 2(1 - \sigma_y)/N$$

Note that for values of ε close to 1, $\sigma_y = 1/2$ and $\bar{\Omega}^y_{\varepsilon,N}$ is uniform. This reflects the fact that we do not require any mesh fitting when the boundary layer is weak.

3.2. Classical finite-difference scheme

Using an upwind finite-difference scheme on the piecewise-uniform mesh $\bar{\Omega}_{\varepsilon,N}$, we approximate the problem (P_ε) by

$$(P_{\varepsilon,N}) \begin{cases} \text{Find } (U_{\varepsilon,N}, V_{\varepsilon,N}) \text{ such that for all } (x_i, y_j) \in \Omega^U_{\varepsilon,N} \\ (-\varepsilon\delta_y^2 + U_{\varepsilon,N}(x_i, y_j)D_x^- + V_{\varepsilon,N}(x_i, y_j)D_y^+)U_{\varepsilon,N}(x_i, y_j) = x_i \\ \text{and for all } (x_i, y_j) \in \Omega^V_{\varepsilon,N} \\ D_x^- U_{\varepsilon,N}(x_i, y_j) + D_y^+ V_{\varepsilon,N}(x_i, y_j) = 0 \\ \text{with the boundary conditions} \\ U_{\varepsilon,N} = 0 \text{ on } \Gamma_{\mathrm{L}} \\ U_{\varepsilon,N} = V_{\varepsilon,N} = 0 \text{ on } \Gamma_{\mathrm{B}}; \quad U_{\varepsilon,N} = \bar{U}^{\mathrm{FS}}_{\varepsilon,N_{\mathrm{FS}}} \text{ on } \Gamma_{\mathrm{T}} \end{cases} \tag{6}$$

where $\Omega^U_{\varepsilon,N} = \{x_i : 0 < i \leqslant N\} \times \{y_j : 0 < j < N\}$, $\Omega^V_{\varepsilon,N} = \{x_i : 0 < i \leqslant N\} \times \{y_j : 0 \leqslant j < N\}$, and for any mesh function $\Phi(x_i, y_j)$,

$$D_y^- \Phi(x_i, y_j) = \frac{\Phi(x_i, y_j) - \Phi(x_i, y_{j-1})}{y_j - y_{j-1}}$$

$$D_y^+ \Phi(x_i, y_j) = \frac{\Phi(x_i, y_{j+1}) - \Phi(x_i, y_j)}{y_{j+1} - y_j}$$

$$\delta_y^2 \Phi(x_i, y_j) = \frac{2}{y_{j+1} - y_{j-1}} \{D_y^+ \Phi(x_i, y_j) - D_y^- \Phi(x_i, y_j)\}$$

D_x^- is defined analogously. The scheme is upwinded provided that $(U_{\varepsilon,N}, V_{\varepsilon,N})$ satisfy $U_{\varepsilon,N}(x_i, y_j) \geqslant 0$, $V_{\varepsilon,N}(x_i, y_j) \leqslant 0$ for all $(x_i, y_j) \in \Omega^U_{\varepsilon,N}$.

The parabolic system of discrete equations in $(P_{\varepsilon,N})$ propagates the numerical solution $(U_{\varepsilon,N}, V_{\varepsilon,N})$ forward from one discrete x-level $\bar{X}_i = \{(x_i, y_j) : 0 \leqslant j \leqslant N\}$ to the next in a step-by-step fashion. Thus, the method of solution involves obtaining $(U_{\varepsilon,N}, V_{\varepsilon,N})$ at the level $\bar{X}_i$ using solution values, namely $U_{\varepsilon,N}$, from the previous level $\bar{X}_{i-1}$ before proceeding to $\bar{X}_{i+1}$. We express this by writing $(P_{\varepsilon,N})$ in the form

$$
\begin{aligned}
&\text{For each } i, 0 < i \leqslant N : \\
&\text{find } (U_{\varepsilon,N}|_{X_i^U}, V_{\varepsilon,N}|_{X_i^V}) \text{ such that for all } y_j \in X_i^U \\
&(-\varepsilon\delta_y^2 + U_{\varepsilon,N}(x_i, y_j)D_x^- + V_{\varepsilon,N}(x_i, y_j)D_y^+)U_{\varepsilon,N}(x_i, y_j) = x_i \\
&\text{and for all } y_j \in X_i^V \\
&D_x^- U_{\varepsilon,N}(x_i, y_j) + D_y^+ V_{\varepsilon,N}(x_i, y_j) = 0 \\
&\text{with the boundary conditions} \\
&U_{\varepsilon,N} \equiv U_{\varepsilon,N}|_{X_{i-1}^U} \text{ on } X_{i-1}^U \text{ with } U_{\varepsilon,N}|_{X_0^U} = 0 \\
&U_{\varepsilon,N} = V_{\varepsilon,N} = 0 \text{ on } \Gamma_B \cap \bar{X}_i; \quad U_{\varepsilon,N} = \bar{U}_{\varepsilon,N_{FS}}^{FS} \text{ on } \Gamma_T \cap \bar{X}_i
\end{aligned}
\tag{7}
$$

where $X_i^U = \{(x_i, y_j) : 0 < j < N\}$, $X_i^V = \{(x_i, y_j) : 0 \leqslant j < N\}$, and $U_{\varepsilon,N}|_{X_i^U} \equiv U_{\varepsilon,N}(x_i, y_j)$ on X_i^U. $V_{\varepsilon,N}|_{X_i^V}$ is defined analogously. At each $\bar{X}_i$ we solve a coupled non-linear system consisting of a two-point boundary value problem for $U_{\varepsilon,N}|_{X_i^U}$ and an initial value problem for $V_{\varepsilon,N}|_{X_i^V}$ using an iterative method.

3.3. Iterative method

The continuation method has been found to be suitable for the solution of non-linear equations in which a small parameter, such as ε, multiplies the highest-order derivative [1]. Applied to (7), the method consists of writing the coupled non-linear system at each $\bar{X}_i$ as a sequence of linear systems parameterized by m, where m runs from 1 to some M_i that is determined using a prescribed residual tolerance. Linearization is obtained by replacing the unknown non-linear coefficients in (7) by known equivalents that have been evaluated at the previous iterate $m-1$. The resulting linear systems are solved for $U_{\varepsilon,N}^m|_{X_i^U}$ using the Thomas algorithm and for $V_{\varepsilon,N}^m|_{X_i^V}$ by integrating along X_i^V.

As an initial guess $(U_{\varepsilon,N}^0|_{X_i^U}, V_{\varepsilon,N}^0|_{X_i^V})$ for the iterative sequence at each $\bar{X}_i$, $2 \leqslant i \leqslant N$, we use the converged solution from the previous level $\bar{X}_{i-1}$ which we write as $(U_{\varepsilon,N}|_{X_{i-1}^U}, V_{\varepsilon,N}|_{X_{i-1}^V}) \equiv (U_{\varepsilon,N}^{M_{i-1}}|_{X_{i-1}^U}, V_{\varepsilon,N}^{M_{i-1}}|_{X_{i-1}^V})$. At $\bar{X}_1$ a different initial guess is required as $V_{\varepsilon,N}|_{X_0^V}$ is unavailable and the use of the zero-valued prescribed boundary condition $U_{\varepsilon,N}|_{X_0^U}$ leads to unnecessarily high iteration counts. We let $(U_{\varepsilon,N}^0|_{X_1^U}, V_{\varepsilon,N}^0|_{X_1^V})$ equal to the inviscid flow solution (u_0, v_0) (defined in (3)) there. The iterative process is continued until

$$
\max\{\|U_{\varepsilon,N}^m - U_{\varepsilon,N}^{m-1}\|_{X_i^U}, \ \|V_{\varepsilon,N}^m - V_{\varepsilon,N}^{m-1}\|_{X_i^V}, \ \|D_y^+ V_{\varepsilon,N}^m - D_y^+ V_{\varepsilon,N}^{m-1}\|_{X_i^V}\} \leqslant \text{tol} \tag{8}
$$

where $\|\cdot\|$ denotes the maximum norm over the indicated domain and tol is a prescribed tolerance. The value of tol is chosen to ensure that for each numerical approximation to the velocity components and their scaled derivatives, the residual error due to the iterative

process is less than the error due to discretization. We include the residual of $D_y^+ V_{\varepsilon,N}$ in this convergence test to guarantee that for small values of ε, $V_{\varepsilon,N}$ correctly captures the lower-order boundary layer that is present in v_ε. The complete iterative method can be written in the form

$$(A_{\varepsilon,N})\begin{cases}
\text{For each } i,\ 0<i\leqslant N: \\
\text{for each } m,\ 1\leqslant m\leqslant M_i: \\
\text{find } (U_{\varepsilon,N}^m|_{X_i^U},\ V_{\varepsilon,N}^m|_{X_i^V}) \text{ such that for all } y_j\in X_i^U \\
(-\varepsilon\delta_y^2 + U_{\varepsilon,N}^{m-1}(x_i,y_j)D_x^- + V_{\varepsilon,N}^{m-1}(x_i,y_j)D_y^+)U_{\varepsilon,N}^m(x_i,y_j) = x_i \\
\text{and for all } y_j\in X_i^V \\
D_x^- U_{\varepsilon,N}^m(x_i,y_j) + D_y^+ V_{\varepsilon,N}^m(x_i,y_j) = 0 \\
\text{with the boundary conditions} \\
U_{\varepsilon,N}^m \equiv U_{\varepsilon,N}|_{X_{i-1}^U} \text{ on } X_{i-1}^U \text{ with } U_{\varepsilon,N}|_{X_0^U} = 0 \\
U_{\varepsilon,N}^m = V_{\varepsilon,N}^m = 0 \text{ on } \Gamma_{\mathrm{B}}\cap\bar{X}_i; \quad U_{\varepsilon,N}^m = \bar{U}_{\varepsilon,N_{\mathrm{FS}}}^{\mathrm{FS}} \text{ on } \Gamma_{\mathrm{T}}\cap\bar{X}_i \\
\text{and the initial guesses} \\
U_{\varepsilon,N}^0|_{X_i^U} = x_i \text{ for } i=1; \quad U_{\varepsilon,N}^0|_{X_i^U} = U_{\varepsilon,N}|_{X_{i-1}^U} \text{ for } 2\leqslant i\leqslant N \\
V_{\varepsilon,N}^0|_{X_i^V} = -y_j \text{ for } i=1; \quad V_{\varepsilon,N}^0|_{X_i^V} = V_{\varepsilon,N}|_{X_{i-1}^V} \text{ for } 2\leqslant i\leqslant N
\end{cases} \tag{9}$$

Note that as the method $(A_{\varepsilon,N})$ does not provide the values $V_{\varepsilon,N}|_{X_0^V}$, we extend $V_{\varepsilon,N}$ back to X_0^V using extrapolation. Upwinding of the scheme is obtained at each $\bar{X}_i$ and for each iteration provided that $(U_{\varepsilon,N}^{m-1}|_{X_i^U}, V_{\varepsilon,N}^{m-1}|_{X_i^V})$ satisfies $U_{\varepsilon,N}^{m-1}|_{X_i^U}\geqslant 0$, $V_{\varepsilon,N}^{m-1}|_{X_i^V}\leqslant 0$.

In the next section, we computationally show that $(A_{\varepsilon,N})$ is an ε-uniform numerical method and use the generated numerical solutions $(U_{\varepsilon,N}, V_{\varepsilon,N})$ to test an experimental technique for the calculation of global ε-uniform error bounds.

4. RESULTS AND ANALYSIS

To examine the ε-uniform properties of the numerical method $(A_{\varepsilon,N})$ it is necessary to generate numerical solutions $(U_{\varepsilon,N}, V_{\varepsilon,N})$ for several values of ε and N. Results are presented in this section for $\varepsilon\in R_\varepsilon = \{2^{-q} : 0\leqslant q\leqslant 20\}$ and $N\in R_N = \{2^r : 5\leqslant r\leqslant 9\}$. The following specific values have been used: in (5) $C=2$, in (8) $\mathrm{tol} = 1.0\times 10^{-07}$ and in (9) $N_{\mathrm{FS}} = 2^{15}$.

Values of $U_{\varepsilon,N}$ and $V_{\varepsilon,N}$ are obtained at the mesh points of $\bar{\Omega}_{\varepsilon,N}$ and used to calculate the discrete first derivatives $D_x^- U_{\varepsilon,N}$, $D_y^+ U_{\varepsilon,N}$, $D_x^- V_{\varepsilon,N}$ and $D_y^+ V_{\varepsilon,N}$. Extrapolation is employed to generate values of the discrete x-derivatives at x_0 and discrete y-derivatives at y_N. We then use bilinear interpolation to interpolate each pointwise numerical approximation from $\bar{\Omega}_{\varepsilon,N}$ to the whole domain $\bar{\Omega}$; the corresponding interpolants are written with an overline, i.e., the interpolant of $U_{\varepsilon,N}$ is written as $\bar{U}_{\varepsilon,N}$. Note that we work with the scaled discrete derivative $\varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}$ rather than $\overline{D_y^+ U_{\varepsilon,N}}$ as the physically important derivative $\partial u_\varepsilon/\partial y$ is of order $\varepsilon^{-1/2}$.

4.1. *Verification that the method $(A_{\varepsilon,N})$ is ε-uniform*

We verify that the method $(A_{\varepsilon,N})$ is ε-uniform by first showing that the numerical solutions $(U_{\varepsilon,N}, V_{\varepsilon,N})$ are computed with an ε-uniform amount of work. We then construct global ε-uniform error bounds of the form $\bar{C}_{\bar{p}} N^{-\bar{p}}$ for the numerical solution components and their scaled discrete first derivatives using the computed Falkner–Skan solution $(\bar{U}^{\mathrm{FS}}_{\varepsilon,2^{15}}, \bar{V}^{\mathrm{FS}}_{\varepsilon,2^{15}})$ as an approximation to the exact solution $(u_\varepsilon, v_\varepsilon)$ of (P_ε).

For each ε and N, the average number of iterations per level $\bar{X}_i$ required for convergence of $(A_{\varepsilon,N})$ is given by

$$I^{\bar{X}_i}_{\varepsilon,N} = \sum_{i=1}^{N} M_i/N$$

where $M_i \equiv M_i(\varepsilon, N)$ is the number of iterations required for convergence of the method at level $\bar{X}_i$, as defined in the previous subsection. Values of $I^{\bar{X}_i}_{\varepsilon,N}$ are presented in Table I along with values of the ε-uniform average number of iterations per level

$$I^{\bar{X}_i}_{N} = \max_{\varepsilon \in R_e} I^{\bar{X}_i}_{\varepsilon,N}$$

In each column the occurrence of $I^{\bar{X}_i}_{N}$ for the lowest value of ε is highlighted. As ε decreases we see that $I^{\bar{X}_i}_{\varepsilon,N}$ stabilizes to an ε-independent limit for each value of N, and so for any value of $\varepsilon \in R_\varepsilon$ the average number of iterations per level will not be larger than $I^{\bar{X}_i}_{N}$. Thus, an ε-uniform amount of computational work is required by $(A_{\varepsilon,N})$ for the generation of $(U_{\varepsilon,N}, V_{\varepsilon,N})$. We also observe that the values of $I^{\bar{X}_i}_{\varepsilon,N}$ and $I^{\bar{X}_i}_{N}$ are practically independent of N. For the mesh with 512 intervals in each direction, the maximum average number of iterations per level is just 13 for all values of $\varepsilon \in R_\varepsilon$.

Table I. Average number of iterations $I^{\bar{X}_i}_{\varepsilon,N}$ and ε-uniform average number of iterations $I^{\bar{X}_i}_{N}$ per level $\bar{X}_i$ required for convergence of $(A_{\varepsilon,N})$ for various values of ε and N.

$\varepsilon \backslash N$	32	64	128	256	512
1	9	10	12	12	13
2^{-02}	10	11	12	13	13
2^{-04}	10	11	12	13	13
2^{-06}	9	10	11	12	12
2^{-08}	8	9	10	11	12
2^{-09}	9	9	10	11	11
2^{-10}	9	9	10	11	11
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
2^{-20}	9	9	10	11	11
$I^{\bar{X}_i}_{N}$	10	11	12	13	13

Global ε-uniform error bounds are calculated with the use of the computed Falkner–Skan solution $(\bar{U}^{\mathrm{FS}}_{\varepsilon,2^{15}}, \bar{V}^{\mathrm{FS}}_{\varepsilon,2^{15}})$ in the following briefly outlined way. An in-depth derivation of the basis for this procedure can be found in Reference [1]. Firstly, for each numerical solution component and discrete derivative, and for all $\varepsilon \in R_\varepsilon$ and $N \in R_N$, we compute the global errors

$$\begin{aligned}\bar{E}_{\varepsilon,N} &= \|\bar{U}_{\varepsilon,N} - \bar{U}^{\mathrm{FS}}_{\varepsilon,2^{15}}\|_{\bar{\Omega}} \\ &= \max\left\{\|U_{\varepsilon,N} - \bar{U}^{\mathrm{FS}}_{\varepsilon,2^{15}}\|_{\bar{\Omega}_{\varepsilon,N}},\ \|\bar{U}_{\varepsilon,N} - U^{\mathrm{FS}}_{\varepsilon,2^{15}}\|_{\bar{\Omega}^x_N \times \{\{y_k : 0 \leqslant k \leqslant 2^{15}\} \cap [0,1]\}}\right\}\end{aligned}$$

defined here for $\bar{U}_{\varepsilon,N}$ but with corresponding definitions for the second component and discrete derivatives. The values of y_k are defined by $y_k = \varepsilon^{1/2}\eta_k$ where $\{\eta_k : 0 \leqslant k \leqslant 2^{15}\}$ are the mesh points of the Falkner–Skan problem. Using $\bar{E}_{\varepsilon,N}$, the global ε-uniform errors

$$\bar{E}_N = \max_{\varepsilon \in R_\varepsilon} \bar{E}_{\varepsilon,N}$$

are formed. We then compute the global ε-uniform orders of local convergence

$$\bar{p}_N = \log_2\left(\frac{\bar{E}_N}{\bar{E}_{2N}}\right)$$

and take the global ε-uniform order of convergence to be

$$\bar{p} = \min_{N:\, N, 2N \in R_N} \bar{p}_N$$

$\bar{p}$ is used to calculate the global ε-uniform error constant

$$\begin{aligned}\bar{C}_{\bar{p}} &= \max_{N \in R_N} \bar{C}_{N,\bar{p}} \\ &= \max_{N \in R_N} \bar{E}_N N^{\bar{p}}\end{aligned}$$

The global ε-uniform error bound is given by $\bar{C}_{\bar{p}} N^{-\bar{p}}$.

Values of $\bar{E}_{\varepsilon,N}$, $\bar{E}_N$ and $\bar{p}_N$ are presented in Table II for $\bar{U}_{\varepsilon,N}$ and in Table III for $\varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}$. In each column the occurrence of $\bar{E}_N$ for the lowest value of ε is highlighted.

Similar results are obtained for the second numerical solution component and the discrete derivatives bar $\overline{D_x^- V_{\varepsilon,N}}$, which approximates the exact derivative $\partial v_\varepsilon/\partial x \equiv 0$ to within the defined tolerance of the numerical method. We therefore do not form an error bound for this discrete derivative. Note that due to the large rate of change of u_ε in the y-direction within the boundary layer, $\partial u_\varepsilon/\partial y$ is the most difficult component to model and hence the worst results are obtained for $\varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}$. In Tables II and III we see that as ε decreases, the values of $\bar{E}_{\varepsilon,N}$ increase and stabilize at a maximum value for each N. Within each row of the table $\bar{E}_{\varepsilon,N}$ decreases as N increases, and similar behaviour is observed for $\bar{E}_N$. $\bar{p}_N$ is at least 0.802 for $\bar{U}_{\varepsilon,N}$ and 0.680 for $\varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}$, with values increasing as N increases.

Table II. Global error $\bar{E}_{\varepsilon,N}$, ε-uniform error $\bar{E}_N$ and ε-uniform order of local convergence $\bar{p}_N$ for $\bar{U}_{\varepsilon,N}$ generated by $(A_{\varepsilon,N})$ for various values of ε and N.

$\varepsilon \backslash N$	32	64	128	256	512
1	9.07×10^{-04}	4.34×10^{-04}	2.13×10^{-04}	1.07×10^{-04}	5.46×10^{-05}
2^{-02}	4.98×10^{-03}	2.46×10^{-03}	1.22×10^{-03}	6.12×10^{-04}	3.07×10^{-04}
2^{-04}	1.19×10^{-02}	5.86×10^{-03}	2.91×10^{-03}	1.45×10^{-03}	7.24×10^{-04}
2^{-06}	2.48×10^{-02}	1.19×10^{-02}	5.87×10^{-03}	2.91×10^{-03}	1.45×10^{-03}
2^{-08}	4.51×10^{-02}	2.48×10^{-02}	1.19×10^{-02}	5.87×10^{-03}	2.91×10^{-03}
2^{-09}	4.51×10^{-02}	2.59×10^{-02}	1.46×10^{-02}	8.18×10^{-03}	4.13×10^{-03}
2^{-10}	4.51×10^{-02}	2.59×10^{-02}	1.46×10^{-02}	8.18×10^{-03}	4.56×10^{-03}
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
2^{-20}	4.51×10^{-02}	2.59×10^{-02}	1.46×10^{-02}	8.18×10^{-03}	4.56×10^{-03}
$\bar{E}_N$	4.51×10^{-02}	2.59×10^{-02}	1.46×10^{-02}	8.18×10^{-03}	4.56×10^{-03}
$\bar{p}_N$	8.02×10^{-01}	8.29×10^{-01}	8.32×10^{-01}	8.43×10^{-01}	

Table III. Global error $\bar{E}_{\varepsilon,N}$, ε-uniform error $\bar{E}_N$ and ε-uniform order of local convergence $\bar{p}_N$ for $\varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}$ generated by $(A_{\varepsilon,N})$ for various values of ε and N.

$\varepsilon \backslash N$	32	64	128	256	512
1	1.79×10^{-02}	8.89×10^{-03}	4.40×10^{-03}	2.14×10^{-03}	1.02×10^{-03}
2^{-02}	3.99×10^{-02}	2.00×10^{-02}	9.98×10^{-03}	4.94×10^{-03}	2.42×10^{-03}
2^{-04}	8.06×10^{-02}	4.08×10^{-02}	2.05×10^{-02}	1.02×10^{-02}	5.06×10^{-03}
2^{-06}	1.57×10^{-01}	8.06×10^{-02}	4.08×10^{-02}	2.05×10^{-02}	1.02×10^{-02}
2^{-08}	2.61×10^{-01}	1.57×10^{-01}	8.06×10^{-02}	4.08×10^{-02}	2.05×10^{-02}
2^{-09}	2.61×10^{-01}	1.63×10^{-01}	9.72×10^{-02}	5.63×10^{-02}	2.89×10^{-02}
2^{-10}	2.61×10^{-01}	1.63×10^{-01}	9.72×10^{-02}	5.63×10^{-02}	3.19×10^{-02}
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
2^{-20}	2.61×10^{-01}	1.63×10^{-01}	9.72×10^{-02}	5.63×10^{-02}	3.19×10^{-02}
$\bar{E}_N$	2.61×10^{-01}	1.63×10^{-01}	9.72×10^{-02}	5.63×10^{-02}	3.19×10^{-02}
$\bar{p}_N$	6.80×10^{-01}	7.44×10^{-01}	7.89×10^{-01}	8.21×10^{-01}	

Global ε-uniform error bounds, valid for all $N \geqslant 32$, for the numerical solutions generated by $(A_{\varepsilon,N})$ and their discrete first derivatives are

$$\max_{\varepsilon \in R_\varepsilon} \|\bar{U}_{\varepsilon,N} - u_\varepsilon\|_{\bar{\Omega}} \leqslant 0.727\, N^{-0.80} \tag{10}$$

$$\max_{\varepsilon \in R_\varepsilon} \|\bar{V}_{\varepsilon,N} - v_\varepsilon\|_{\bar{\Omega}} \leqslant 0.685\, N^{-1.0} \tag{11}$$

Table IV. Global error bound $\bar{C}_{\bar{p}}N^{-\bar{p}}$ for $\bar{U}_{\varepsilon,N}$ generated by $(A_{\varepsilon,N})$ for various values of N.

$\varepsilon\backslash N$	32	64	128	256	512
$\bar{C}_{\bar{p}}N^{-\bar{p}}$	4.51×10^{-02}	2.59×10^{-02}	1.48×10^{-02}	8.50×10^{-03}	4.88×10^{-03}

Table V. Global error bound $\bar{C}_{\bar{p}}N^{-\bar{p}}$ for $\varepsilon^{1/2}\overline{D_y^+U_{\varepsilon,N}}$ generated by $(A_{\varepsilon,N})$ for various values of N.

$\varepsilon\backslash N$	32	64	128	256	512
$\bar{C}_{\bar{p}}N^{-\bar{p}}$	2.61×10^{-01}	1.63×10^{-01}	1.02×10^{-01}	6.35×10^{-02}	3.96×10^{-02}

$$\max_{\varepsilon\in R_\varepsilon}\left\|\overline{D_x^-U_{\varepsilon,N}}-\frac{\partial u_\varepsilon}{\partial x}\right\|_{\bar{\Omega}}\leqslant 0.727\,N^{-0.80} \tag{12}$$

$$\max_{\varepsilon\in R_\varepsilon}\left\|\varepsilon^{1/2}\left(\overline{D_y^+U_{\varepsilon,N}}-\frac{\partial u_\varepsilon}{\partial y}\right)\right\|_{\bar{\Omega}}\leqslant 2.75\,N^{-0.68} \tag{13}$$

$$\max_{\varepsilon\in R_\varepsilon}\left\|\overline{D_y^+V_{\varepsilon,N}}-\frac{\partial v_\varepsilon}{\partial y}\right\|_{\bar{\Omega}}\leqslant 0.727\,N^{-0.80} \tag{14}$$

which provide verification that $(A_{\varepsilon,N})$ is an ε-uniform method. In each case the global ε-uniform error constant does not exceed 2.75 and the global ε-uniform order of convergence is at least 0.68. In Tables IV and V we evaluate bounds (10) and (13), respectively, for all $N\in R_N$.

Comparison with Tables II and III reveals that the error bounds $\bar{C}_{\bar{p}}N^{-\bar{p}}$ are equal to or slightly overestimate the error values $\bar{E}_N$ by a factor of at most 1.07 for $\bar{U}_{\varepsilon,N}$ and 1.24 for $\varepsilon^{1/2}\overline{D_y^+U_{\varepsilon,N}}$.

The practical uses of bounds (10)–(14) include being able to determine the number of mesh intervals N that must be used to obtain numerical approximations with a required level of guaranteed accuracy. For example, to generate a numerical approximation to u_ε for any value of $\varepsilon\in R_\varepsilon$ with a global error of less than 0.01, we calculate from (10) that N must satisfy $N\geqslant 213$. Such information is of great practical benefit, particularly when approximating important physical quantities. Consider for example the dimensionless shearing stress at the plate $\tau_{y=0,\varepsilon}(x)$ which is defined by

$$\tau_{y=0,\varepsilon}(x)=\varepsilon\left(\frac{\partial u_\varepsilon}{\partial y}\right)_{y=0}$$

[6]. As mentioned at the beginning of this section, $\partial u_\varepsilon/\partial y$ is of order $\varepsilon^{-1/2}$ and so a suitably scaled dimensionless shearing stress is $\varepsilon^{-1/2}\tau_{y=0,\varepsilon}$. Numerical experiments reveal that the value of $\bar{E}_{\varepsilon,N}$ for $\varepsilon^{1/2}\overline{D_y^+U_{\varepsilon,N}}$ is attained at $y=0$. We can therefore use the bound (13) directly to write a global ε-uniform error bound for the numerical approximations

$\varepsilon^{-1/2}\bar{\mathscr{T}}_{y=0,\varepsilon,N} = \varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}|_{y=0}$ to the scaled dimensionless shearing stress

$$\max_{\varepsilon\in R_\varepsilon} \|\varepsilon^{-1/2}(\bar{\mathscr{T}}_{y=0,\varepsilon,N} - \tau_{y=0,\varepsilon})\|_{\bar{\Omega}} \leqslant 2.75N^{-0.68} \tag{15}$$

The level of accuracy of $\varepsilon^{-1/2}\bar{\mathscr{T}}_{y=0,\varepsilon,N}$ can be controlled as required using N for all values of $\varepsilon \in R_\varepsilon$.

4.2. Test of an experimental technique for computing global ε-uniform error bounds

The technique employed in the previous subsection for verifying that a numerical method is ε-uniform can only be applied when an exact or reference solution of the flow problem is available. This is usually not the case. A generally applicable experimental technique for computing realistic ε-uniform error bounds was developed and successfully applied in Reference [1] and further examined in Reference [7]. Here, we apply an extended version of this technique to the numerical approximations generated by the method $(A_{\varepsilon,N})$ and we test the resulting experimental global ε-uniform error bounds by comparison with the global ε-uniform error values obtained with the use of the reference solutions.

The experimental technique proceeds in the following way. For each numerical solution component and discrete derivative, and for all $\varepsilon \in R_\varepsilon$ and N such that $N, 2N \in R_N$, we calculate the experimental global two-mesh differences

$$\bar{D}^*_{\varepsilon,N} = \|\bar{U}_{\varepsilon,N} - \bar{U}_{\varepsilon,2N}\|_{\bar{\Omega}} = \max\{\|U_{\varepsilon,N} - \bar{U}_{\varepsilon,2N}\|_{\bar{\Omega}_{\varepsilon,N}}, \|\bar{U}_{\varepsilon,N} - U_{\varepsilon,2N}\|_{\bar{\Omega}_{\varepsilon,2N}}\}$$

defined here for $\bar{U}_{\varepsilon,N}$ but with analogous definitions for the second component and discrete derivatives. The experimental global ε-uniform two-mesh differences

$$\bar{D}^*_N = \max_{\varepsilon\in R_\varepsilon} \bar{D}^*_{\varepsilon,N}$$

are formed and used to compute the experimental global ε-uniform orders of local convergence

$$\bar{p}^*_N = \log_2\left(\frac{\bar{D}^*_N}{\bar{D}^*_{2N}}\right)$$

We take the experimental global ε-uniform order of convergence to be

$$\bar{p}^* = \min_{N:N,2N,4N\in R_N} \bar{p}^*_N$$

and calculate the corresponding experimental global ε-uniform error constant

$$\bar{C}^*_{\bar{p}^*} = \max_{N:N,2N\in R_N} \bar{C}^*_{N,\bar{p}^*} = \max_{N:N,2N\in R_N}\left(\frac{\bar{D}^*_N N^{\bar{p}^*}}{1-2^{-\bar{p}^*}}\right)$$

The experimental global ε-uniform error bound is then given by $\bar{C}^*_{\bar{p}^*}N^{-\bar{p}^*}$.

Values of $\bar{D}^*_N$ and $\bar{p}^*_N$ are presented in Table VI for $\bar{U}_{\varepsilon,N}$ and in Table VII for $\varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}$. Analogous results are obtained for the second numerical solution component and the discrete derivatives bar $\overline{D_x^- V_{\varepsilon,N}}$, for reasons discussed in the previous subsection. As N increases we see that the values of $\bar{D}^*_N$ decrease in both cases. The irregular behaviour of $\bar{p}^*_N$ is found to be

Table VI. Experimental global two-mesh difference $\bar{D}_N^*$ and order of local convergence $\bar{p}_N^*$ for $\bar{U}_{\varepsilon,N}$ generated by $(A_{\varepsilon,N})$ for various values of N.

$\varepsilon\backslash N$	32	64	128	256
$\bar{D}_N^*$	2.60×10^{-02}	1.41×10^{-02}	6.81×10^{-03}	4.11×10^{-03}
$\bar{p}_N^*$	8.80×10^{-01}	$1.05 \times 10^{+00}$	7.30×10^{-01}	

Table VII. Experimental global two-mesh difference $\bar{D}_N^*$ and order of local convergence $\bar{p}_N^*$ for $\varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}$ generated by $(A_{\varepsilon,N})$ for various values of N.

$\varepsilon\backslash N$	32	64	128	256
$\bar{D}_N^*$	1.04×10^{-01}	7.64×10^{-02}	4.10×10^{-02}	2.74×10^{-02}
$\bar{p}_N^*$	4.47×10^{-01}	9.00×10^{-01}	5.81×10^{-01}	

Table VIII. Experimental global error bound $\bar{C}_{\bar{p}^*}^* N^{-\bar{p}^*}$ for $\bar{U}_{\varepsilon,N}$ generated by $(A_{\varepsilon,N})$ for various values of N.

$\varepsilon\backslash N$	32	64	128	256
$\bar{C}_{\bar{p}^*}^* N^{-\bar{p}^*}$	6.54×10^{-02}	3.94×10^{-02}	2.38×10^{-02}	1.43×10^{-02}

due to the interaction of the closely valued errors in the numerical approximations generated using N intervals and $2N$ intervals. $\bar{p}_N^*$ is at least 0.730 for $\bar{U}_{\varepsilon,N}$ and 0.447 for $\varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}$.

The experimental global ε-uniform error bounds are

$$\max_{\varepsilon \in R_\varepsilon} \|\bar{U}_{\varepsilon,N} - u_\varepsilon\|_{\bar{\Omega}} \leqslant 0.821\, N^{-0.73} \tag{16}$$

$$\max_{\varepsilon \in R_\varepsilon} \|\bar{V}_{\varepsilon,N} - v_\varepsilon\|_{\bar{\Omega}} \leqslant 0.704\, N^{-1.0} \tag{17}$$

$$\max_{\varepsilon \in R_\varepsilon} \left\| \overline{D_x^- U_{\varepsilon,N}} - \frac{\partial u_\varepsilon}{\partial x} \right\|_{\bar{\Omega}} \leqslant 0.821\, N^{-0.73} \tag{18}$$

$$\max_{\varepsilon \in R_\varepsilon} \left\| \varepsilon^{1/2} \left(\overline{D_y^+ U_{\varepsilon,N}} - \frac{\partial u_\varepsilon}{\partial y} \right) \right\|_{\bar{\Omega}} \leqslant 1.84\, N^{-0.45} \tag{19}$$

$$\max_{\varepsilon \in R_\varepsilon} \left\| \overline{D_y^+ V_{\varepsilon,N}} - \frac{\partial v_\varepsilon}{\partial y} \right\|_{\bar{\Omega}} \leqslant 0.821\, N^{-0.73} \tag{20}$$

Bounds (16) and (19) are evaluated in Tables VIII and IX, respectively, for each value of N such that $N, 2N \in R_N$. By comparison with Tables II and III we find that the bounds generated by the experimental technique overestimate the error values $\bar{E}_N$ by a factor of at most 1.75 for $\bar{U}_{\varepsilon,N}$ and 2.74 for $\varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}$. We conclude that the experimental technique provides realistic global ε-uniform error bounds for the numerical approximations generated by $(A_{\varepsilon,N})$.

Table IX. Experimental global error bound $\bar{C}^*_{\bar{p}^*} N^{-\bar{p}^*}$ for $\varepsilon^{1/2}\overline{D_y^+ U_{\varepsilon,N}}$ generated by $(A_{\varepsilon,N})$ for various values of N.

$\varepsilon \backslash N$	32	64	128	256
$\bar{C}^*_{\bar{p}^*} N^{-\bar{p}^*}$	3.91×10^{-01}	2.87×10^{-01}	2.11×10^{-01}	1.54×10^{-01}

5. CONCLUSIONS

A recently developed ε-uniform numerical method for the solution of the Prandtl equations was examined for applicability to the stagnation line flow problem. The method was shown to generate ε-uniformly convergent numerical approximations in a domain inclusive of the stagnation line. Global ε-uniform error bounds, valid for all $N \geqslant 32$, were constructed for the numerical solution components, their appropriately scaled discrete first derivatives and the scaled dimensionless shearing stress. We found that in each case the error constant $\bar{C}_{\bar{p}}$ did not exceed 2.75 and the order of convergence $\bar{p}$ was not less than 0.68. We demonstrated the use of these error bounds for determining the minimum value of N that must be used to obtain numerical approximations with a required level of guaranteed accuracy independently of the value of ε. Furthermore, the average number of iterations per level required for convergence of the method was shown to be ε-uniform and practically independent of N. For the mesh with 512 intervals in each direction, we found that the maximum average number of iterations per level was just 13 for all values of $\varepsilon \in R_\varepsilon$.

Global ε-uniform error bounds were constructed using an experimental technique. They were shown to be realistic upper bounds for the error values obtained using the reference solutions, thereby providing reliable information regarding the ε-uniform convergence of the generated numerical approximations.

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